



Alternative Variational Iteration Method for Solving Time-Fractal-Fractional Partial Differential Equations

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Abstract

The alternative variational iteration method (AVIM) is applied to solve time fractal-fractional partial differential equations. We investigate the one-dimensional time fractional diffusion-wave equation and a generalized time-fractional partial differential equation (PDE) with variable coefficients, modelling wave propagation in mathematical physics under the new generalized Caputo-type fractal-fractional derivative. The validity and efficiency of the method, as well as the effect of the fractal dimension, are demonstrated through numerical test examples. Graphical representations of numerical results for different parameter values are also provided. The acquired numerical results employing the proposed method are compared equally with those gotten using the traditional variational iteration method and AVIM with fractional derivatives only, as found in the literature. The results indicate that the proposed approach has the advantage of being able to solve the problem with fewer iterations due to the extra parameter and fractal dimension utilized in the fractal-fractional derivative.

Keywords: fractal-fractional operator; new generalized Caputo derivative; alternative variational iterative method; Fornberg–Whitham equation with variable coefficients; diffusion equation.

1 Introduction

Recently, consideration for fractional order partial differential equations (FPDEs) has considerably increased due to their great importance and suitability in modelling the dynamical behaviours of problems and processes in different science and engineering fields. Like time-fractional oscillation and relaxation phenomena, time-fractional wave and diffusive behaviours [37], dynamic processes within self-similar and porous frameworks, atom-field interactions within photonic crystals, viscoelastic phenomena, electrical network dynamics, signal processing applications, fractal geometric concepts, chemical processes and biological systems [11], among other disciplines within science and engineering [52, 17]. Several fractional derivative (FD) as well as integral operators that differ in terms of their properties have been defined and employed widely to model real-life problems in literature [43, 30]. One of the most widely used fractional derivatives is the Caputo and Caputo-type fractional derivatives due to their ability to handle conventional initial and boundary conditions (BCs) in the representation of physical problems. For instance, Nawaz et al. [45] applied the Caputo derivative to model the transmission of Tuberculosis. While, the generalized Caputo derivative [36] shares the same characteristics as Caputo derivative [35, 48], the new generalized Caputo type fractional derivative employed [21] is equipped with an extra parameter not found in other fractional derivatives, making it a more versatile version. The continuous emergence of new fractional operators may be attributed to the occurrence of new problems that reflect various real-world scenarios.

Worthy of note is that obtaining exact solutions to many FPDEs is often difficult, if not impossible. Therefore, approximate analytical and numerical methods are being developed to provide accurate approximations to FPDEs. Bouazza et al. [13] applied the analytical approach known as Kuratowski metric of non-compactness (MNC) to investigate the existence and stability of solutions to boundary value problem of Riemann–Liouville fractional differential equations of variable order. Furthermore, robust semi-analytic methods, such as the homotopy perturbation method (HPM) [16, 32], Adomian decomposition method (ADM) [34, 49], homotopy analysis method (HAM) [59, 23], reproducing kernel space method [24, 60], and generalized differential transform method (DTM) [44, 41], as well as numerical methods including finite difference method [42, 57], Adams–Bashforth–Moulton method [18], and operational matrix methods [28, 3], among others, have been devised to solve FPDEs. He [25] proposed the variational iteration method (VIM) for solving non-linear differential equations. Subsequently, several modifications of the VIM have been suggested and applied to a variety of problems with the aim of improving convergence speed and suitability [25, 1].

In this study, we consider the fractional diffusion-wave equation [53] and the generalized non-linear time-fractional PDE with variable coefficients [5] as given in (1) and (2) respectively below,

$$\frac{\partial^\alpha \varphi}{\partial \varsigma^\alpha} = c \frac{\partial^2 \varphi}{\partial x^2}, \quad (1)$$

with the initial and BCs,

$$\varphi(0, \varsigma) = \varphi(L, \varsigma) = 0, \quad \varphi(x, 0) = \phi(x), \quad \frac{\partial \varphi(x, 0)}{\partial x} = \sigma(x), \quad 0 \leq x \leq L, \quad \varsigma > 0,$$

where α is the fractional order and the coefficient c is a constant known as the diffusion constant, x and ς are the space and time variables respectively. For $\alpha = 1$, (1) represents a diffusion equation. If $\alpha = 2$, it leads to a wave equation,

$$\begin{aligned} &\mathfrak{D}_\tau^\alpha \phi(\kappa, \tau) - \mu(\kappa) \phi_{\kappa\kappa\tau}(\kappa, \tau) + 2\varphi(\kappa) \phi_\kappa(\kappa, \tau) + \omega(\kappa) \phi(\kappa, \tau) \phi_\kappa(\kappa, \tau) \\ &- \theta(\kappa) \phi(\kappa, \tau) \phi_{\kappa\kappa\kappa}(\kappa, \tau) - \sigma(\kappa) \phi_\kappa(\kappa, \tau) \phi_{\kappa\kappa}(\kappa, \tau) = 0, \quad \ell < \kappa < \mathcal{L}, \quad \ell, \mathcal{L} \in \mathbb{R}, \end{aligned} \quad (2)$$

with the initial and BCs,

$$\phi(\kappa, 0) = \psi(\kappa), \phi(\ell\tau) = 0, \quad \phi(\mathcal{L}, \tau) = 0, \quad \tau \geq 0,$$

where $\mu(\kappa)$, $\varphi(\kappa)$, $\omega(\kappa)$, $\theta(\kappa)$, and $\sigma(\kappa)$ are the variable coefficients.

Equation (2) represents a more general model for wave propagation, and for specific choices of the variable coefficients, we obtain a variety of interesting and important PDEs in literature. For instance, when $\alpha = 1$, $\varphi = \frac{1}{2}$, $\mu = \omega = \theta = 1$ and $\sigma = 3$, the very famous and well investigated Fornberg–Whitham equation [22] is obtained as,

$$\phi_\tau - \phi_{\kappa\kappa\tau} + \phi_\kappa + \phi\phi_\kappa - \phi\phi_{\kappa\kappa} - 3\phi_\kappa\phi_{\kappa\kappa} = 0, \quad (3)$$

while $\alpha = 1$, $\mu = \varphi = \sigma = 0$, $\omega = 6$ and $\theta = -1$ leads to the Korteweg–de Vries (KdV) equation [40],

$$\phi_\tau + 6\phi\phi_\kappa + \phi\phi_{\kappa\kappa} = 0, \quad (4)$$

and $\alpha = 1$, $\mu = \sigma = 1$, $\theta = -2$ and $\omega = 3$ gives the Camassa–Holm Equation in [22] as,

$$\phi_\tau - \phi_{\kappa\kappa\tau} + 2\varphi\phi_\kappa + 3\phi\phi_\kappa - 2\phi\phi_{\kappa\kappa} - \phi_\kappa\phi_{\kappa\kappa} = 0. \quad (5)$$

Over the past years, various methods have been employed to approximate solutions for different instances of (1) and (2). Donatelli et al. [19] investigated the diffusion equation, which is prevalent in numerous applications and comprises of fractional derivatives and variable coefficients. Heydari et al. [27] developed a wavelet method for solving time-fractional fractional diffusion equations while Heydari et al. [29] studied the variable-order nonlinear fractional diffusion-wave equations. Additional approaches applied to diffusion equations can be found in [31, 53]. Odibat [50] introduced the alternative variational iterative method (AVIM) for the solution of linear and nonlinear differential equation models. Sakar and Ergören [54] proposed another AVIM technique for obtaining solutions to the time-fractional Fornberg–Whitham equation, which involves the Caputo derivative.

Singh and Kumar [56] utilized the AVIM method to derive solutions for time fractional partial differential equations with proportional delay. Boutarfa et al. [14] explored three types of Fornberg–Whitham equations: original, modified, and time-fractional using the reproducing kernel Hilbert space method. Alsdrani [5] obtained approximate solutions for a third-order time-fractional Fornberg–Whitham equation involving variable coefficients by employing three numerical techniques: VIM, ADM and HAM. Heydari and Zhagharian [28] proposed a technique to address fractional Fornberg–Whitham equations with variable order. Noor et al. [47] proposed the VIM with the Elzaki transform to approximate solutions to fractional diffusion equation. The study demonstrated the effectiveness of combining the VIM and the Elzaki transform in handling fractional differential equations. Kumar et al. [39] applied the Elzaki residual power series method to obtain semi-analytic approximation of time-fractional nonlinear equations occurring in magneto-acoustic waves. Ullah et al. [58] combined Laplace transform and Picard’s iterative scheme for obtaining optimal semi-analytical solutions of time-fractional evolution equations.

Alhejaili et al. [4] employed Yang homotopy perturbation method and the Yang transform decomposition method to solve fractional forced Korteweg–de Vries equation arising in fluids and plasmas. Nazari-Golshan [46] proposed a traveling wave solution for solving non-linear space-time fractional Fornberg–Whitham equation. Iqbal et al. [33] proposed the optimal auxiliary function method for solving the fractional-order Fornberg–Whitham problem in the sense

of the Caputo fractional operator. The Homotopy perturbation transform method was applied to solve the time-fractional FW equation with the Caputo-Fabrizio fractional derivative [12]. The authors discussed the existence and uniqueness of this equation using fixed point theory. Enyi et al. [20] developed a method called the q -Homotopy analysis conformable fractional Laplace transform method for analytical and numerical study of a space-time conformable fractional Fornberg-Whitham equation. He and Rui [26] presented an extended separation method of semi-fixed variables for analytical solution of the time-fractional Fornberg-Whitham equation, the time-fractional porous medium equation and the time-fractional Hunter-Saxton equation, among other methods that have been deployed to solving the model fractional PDEs under study.

While many dynamical behaviors of systems modeled by partial differential equations (PDEs) have been simulated using fractional PDEs, with several semi-analytical and numerical techniques in vogue for solving these fractional PDEs, a more general approach that has recently garnered significant interest is the utilization of fractal-fractional derivatives in place of fractional derivatives in partial differential equation models. This generalization of fractional derivatives with an additional fractal dimension provides a more general way to represent more complex systems and physical processes, addressing any potential limitations of the use of fractional derivatives in modelling problems that have complex patterns that exhibit self-similarity across different scales [15, 38]. Atangana [6] proposed a novel derivative with fractional and fractal parameters, including power-law, exponential law, and Mittag-Leffler law. This derivative has been employed by several researchers to model various fractal-fractional problems and real-life systems including mathematical modelling of convective fluid motion in rotating cavity [2], groundwater dynamics [7], chaotic system dynamics based on circuit design [38] and modelling attractors of chaotic dynamical systems [8].

The new fractal-fractional derivative has been employed with different techniques such as reproducing kernel Hilbert space method [9], deep learning via Hopfield neural networks [10]. Partohaghghi [51] investigated a variant of the fractal-fractional diffusion equation using the operational matrix of Chebyshev cardinal functions. More recently, Shloof et al. [55] introduced the new generalized Caputo-type fractal-fractional derivative (NGCFFD), incorporating another parameter (fractal dimension) in to the generalized Caputo fractional derivative. The authors applied this operator to solve fractal-fractional differential equations (FFDEs). Due to the limited research on fractal-fractional derivatives compared to traditional fractional derivatives, ongoing efforts are focused on developing and analyzing semi-analytical and numerical methods for solving fractal-fractional problems. As much as we are aware, the AVIM method has not been explored in conjunction with the NGCFFD for solving (1) and (2) in the literature.

In light of the foregoing review, the present study expands upon the application of AVIM as introduced in [50] to address the solution of time-FPDEs. We establish the model FPDEs in terms of fractal-fractional derivatives and devise an iterative procedure based on the AVIM technique, employing the new generalized Caputo derivative, to solve FPDEs as outlined in [55]. The usefulness of the developed method is evaluated by comparing the numerical results got using the newly proposed approach with those obtained using methods discussed in the literature. The numerical solutions obtained via the suggested technique are found to align with those obtained using alternative approaches. The method offers good accuracy with fewer expansion terms, resulting in lower computational cost compared to conventional techniques. It effectively addresses non-linearities without requiring linearization and preserves analytical structure, which can give a deeper understanding of the behaviour of the solution.

We organize this article as thus: Section 2 enumerates key results and definitions from the literature. Section 3 introduces the procedure for the alternative variational iterative method (AVIM), followed by the derivation of the AVIM solution method for (1) and (2) in Section 4. Section 5

provides numerical examples, and Section 6 analyzes the results and offers concluding remarks.

2 Preliminaries

Here, we offer fundamental definitions and properties from fractal and fractional calculus theory, essential for our current investigation.

Definition 2.1. [48] The generalized fractional integral of a function $u(x, t)$ is denoted by $I_{a+}^{\alpha, \beta} u(x, t)$ and defined as,

$$I_{a+}^{\alpha, \beta} u(x, t) = \frac{\beta^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \tau^{\beta-1} (t^\beta - \tau^\beta)^{\alpha-1} d\tau, \quad \beta > 0, \quad \text{and} \quad t > a. \quad (6)$$

Definition 2.2. [48] Suppose $u(x, t)$ is continuous on (a, b) , then the generalized fractal-fractional integral, $I_{a+}^{\alpha, \beta, \gamma} u(x, t)$ of order α is defined as,

$$u(x, t) = \frac{\gamma \beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\beta+\gamma-2} (t^\beta - \tau^\beta)^{\alpha-1} u(x, \tau) d\tau. \quad (7)$$

Definition 2.3. [55] Let $\varsigma - 1 < \alpha \leq \varsigma$ and $\varsigma \in \mathbb{N}$. The Caputo fractional derivative of order $\alpha > 0$ of the function $u(x, t)$ is defined as,

$${}^C \mathfrak{D}_{a+}^{\alpha} u(x, t) = \frac{1}{\Gamma(\varsigma - \alpha)} \int_a^t (t - \tau)^{\varsigma - \alpha - 1} u^{(\varsigma)}(x, \tau) d\tau, \quad t > a. \quad (8)$$

Definition 2.4. [48] Let $u(x, t) \in C^\eta([a, b])$. The new generalized Caputo-type FD having order $\alpha > 0$ is defined as,

$${}^C \mathfrak{D}_{a+}^{\alpha, \beta} u(x, t) = \frac{\beta^{\alpha-\eta+1}}{\Gamma(\eta - \alpha)} \int_a^t \tau^{\beta-1} (t^\beta - \tau^\beta)^{\eta-\alpha-1} \left(t^{1-\beta} \frac{d}{dt} \right)^\eta u(x, \tau) d\tau, \quad (9)$$

where $t > a$, $\beta > 0$ and $\eta - 1 < \alpha \leq \eta$, $\eta = \lceil \alpha \rceil$ and the NGCD has these properties, ${}^C \mathfrak{D}_{a+}^{\alpha, \beta} C = 0$, C is a constant and for $\varsigma - 1 < \alpha \leq \varsigma$, $k > \varsigma - 1$, $k \notin \mathbb{N}$,

$${}^C \mathfrak{D}_{a+}^{\alpha, \beta} (t^\beta - \tau^\beta) = \begin{cases} \beta^\alpha \frac{\Gamma(k+1)}{\Gamma(k - \alpha + 1)} (t^\beta - \tau^\beta)^{k-\alpha}, & k \in \mathbb{N}_0, \quad \text{and} \quad k \geq \lceil \alpha \rceil, \quad \text{or} \quad k > \lfloor \alpha \rfloor, \\ 0, & k \in \mathbb{N}_0, \quad \text{and} \quad \text{or} \quad k < \lceil \alpha \rceil. \end{cases} \quad (10)$$

Definition 2.5. [55] Suppose $u(x, t)$ is continuous and fractal differentiable of order γ on (a, b) and $0 < \varsigma - 1 < \gamma$, $\alpha \leq \varsigma$, then the new generalized Caputo-type fractal-fractional derivative (NGCFFD) of $u(x, t)$ of order α and β is denoted by ${}^C \mathfrak{D}_{a+}^{\alpha, \beta, \gamma} u(x, t)$ and defined as,

$${}^C \mathfrak{D}_{a+}^{\alpha, \beta, \gamma} u(x, t) = \frac{\beta^{\alpha-\varsigma+1}}{\Gamma(\varsigma - \alpha)} \int_a^t \tau^{\beta-1} (t^\beta - \tau^\beta)^{\varsigma-\alpha-1} \left(t^{1-\beta} \frac{d}{dt} \right)^\varsigma u(x, \tau) d\tau, \quad (11)$$

where $\frac{du}{dt^\gamma} = \lim_{t \rightarrow \tau} \frac{u(x, t) - v(x, \tau)}{t^\beta - \tau^\beta}$.

Corollary 2.1. [55] Let $\alpha, \gamma \in (0, 1)$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$. Then, we have

$${}^C \mathfrak{D}_{a+}^{\alpha, \beta, \gamma} t^k = \frac{\beta^\alpha \Gamma\left(\frac{k}{\beta} + 1\right)}{\gamma \Gamma\left(\frac{k}{\beta} - \alpha + 1\right)} t^{\frac{k}{\beta} - \alpha - \gamma + 1}. \quad (12)$$

Theorem 2.1. [35] Let $\varsigma - 1 < \alpha \leq \varsigma$, $\varsigma \in \mathbb{N}$ and $\alpha > 0$. Then, the derivative operator ${}^C\mathfrak{D}^{\alpha,\beta,\gamma}$ satisfies the following property,

$$I_{a+}^{\alpha,\beta,\gamma} {}^C\mathfrak{D}^{\alpha,\beta,\gamma} u(x, t) = u(x, t) - \sum_{k=0}^{\varsigma} \frac{(-1)^k (\gamma^k u)(a)}{k!} \left(\frac{a^\beta - t^\beta}{\beta} \right)^k.$$

Particularly, for $0 < \alpha \leq 1$, we have that,

$$I_{a+}^{\alpha,\beta,\gamma} {}^C\mathfrak{D}^{\alpha,\beta,\gamma} u(x, t) = u(x, t) - u(x, a).$$

3 Alternative Variational Iteration Method (AVIM) Procedure

This section presents an overview of the overall framework of the alternative variational iterative technique for attaining approximate solutions to fractional PDEs, followed by a discussion of its usage in fractal-fractional PDEs.

Consider the non-linear FPDE,

$${}^C\mathfrak{D}_\tau^\alpha \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) = g(\kappa, \tau), \quad (13)$$

where $\varsigma - 1 < \alpha \leq \varsigma$, $\varsigma \in \mathbb{N}$. L and N refer to linear and non-linear operators, respectively. g is a specified analytic function called the source term and ${}^C\mathfrak{D}_t^\alpha$ is the Caputo fractional derivative of order α . The initial conditions for (13) are given as,

$$\phi^k(\kappa, 0) = f_k(\kappa), \quad k = 0, 1, \dots, \varsigma - 1.$$

The general framework of fractional VIM is aimed at obtaining a series solution to (13) in the form,

$$\phi(\kappa, \tau) = \lim_{k \rightarrow \infty} \phi_k(\kappa, \tau), \quad k = 0, 1, \dots, \varsigma - 1, \quad (14)$$

by constructing the correctional function,

$$\phi_{k+1}(\kappa, \tau) = \phi_k(\kappa, \tau) + J_\tau^\alpha \left[\lambda(s) \left({}^C\mathfrak{D}_\tau^\alpha \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) - g(\kappa, \tau) \right) \right], \quad (15)$$

to (13), where J_τ^α denotes the Riemann–Liouville fractional integral and λ represents Lagrange multiplier that is determined by variational theory. Generally, λ is given by,

$$\lambda = \frac{(-1)^m}{(m-1)!} (s - \tau)^{m-1}, \quad m \geq 1. \quad (16)$$

This correctional function is formulated from a variational functional based on the residual of the given fractional PDE adapted to the specific type of fractional derivative, with a Lagrange multiplier introduced in the formulation and an associated fractional integral.

Substituting λ into (15) gives the general VIM iteration formula,

$$\phi_{k+1}(\kappa, \tau) = \phi_k(\kappa, \tau) + J_\tau^\alpha \left[\frac{(-1)^m}{(m-1)!} (s - \tau)^{m-1} \left({}^C\mathfrak{D}_\tau^\alpha \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) - g(\kappa, \tau) \right) \right]. \quad (17)$$

Now, according to [50], the AVIM is deduced from the general VIM iteration formula given in (17) with the following iterative procedure.

From (17), we define the operator,

$$\mathcal{A}[\phi(\kappa, \tau)] = J_\tau^\alpha \left[\frac{(-1)^m}{(m-1)!} (s-\tau)^{m-1} \left({}^C\mathfrak{D}_\tau^\alpha \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) - g(\kappa, \tau) \right) \right], \quad (18)$$

and obtain the iterative components of (18) as follows,

$$\begin{cases} \eta_0 = \phi(\kappa, 0), \\ \eta_1 = \mathcal{A}[\eta_0], \\ \eta_2 = \mathcal{A}[\eta_0 + \eta_1], \\ \vdots \\ \eta_{k+1} = \mathcal{A}[\eta_0 + \eta_1 + \cdots + \eta_k], \end{cases} \quad (19)$$

where $\phi(\kappa, 0)$ is the initial condition to the given FPDE.

$$\text{Thus, we have } \phi(\kappa, \tau) = \lim_{k \rightarrow \infty} \phi_k(\kappa, \tau) = \sum_{k=0}^{\infty} \eta_k.$$

Consequently, the solution to the FPDE in (13) using the AVIM is given by,

$$\phi(\kappa, \tau) = \sum_{k=0}^{\infty} \eta_k(\kappa, \tau). \quad (20)$$

4 AVIM for Fractal-fractional Partial Differential Equations (FFAVIM)

Here, we provide the contribution of this current research: the concept of the fractal-fractional AVIM method. To start with, we apply the NGCFFD to our model FPDEs. Subsequently, we derive the generalized Caputo-type fractal-fractional integral of a power function, which can be generalized to other types of functions.

Consider the non-linear fractal-fractional PDE for $m-1 < \alpha \leq m, m \in \mathbb{N}$,

$${}^C\mathfrak{D}_\tau^{\alpha, \beta, \gamma} \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) = g(\kappa, \tau), \quad (21)$$

where $\mathfrak{D}_t^{\alpha, \beta, \gamma}$ is the NGCFFD, L and N respectively represent linear and non-linear operators, g is a given function called the source term with initial conditions given as,

$$\phi^k(\kappa, 0) = f_k(\kappa), \quad k = 0, 1, \dots, \alpha - 1.$$

To obtain the solution to (21) by the proposed FFAVIM method, we define the FFAVIM iterative operator $\mathcal{A}[\phi]$, with the aid of (18) as follows,

$$\mathcal{A}[\phi] = I_\tau^{\alpha, \beta, \gamma} \left[\frac{(-1)^m}{(m-1)!} (s-\tau)^{m-1} \left({}^C\mathfrak{D}_\tau^{\alpha, \beta, \gamma} \phi(\kappa, \tau) + L\phi(\kappa, \tau) + N\phi(\kappa, \tau) - g(\kappa, \tau) \right) \right], \quad (22)$$

where $I_t^{\alpha, \beta, \gamma}$ is the new generalized fractal-fractional integral as presented in Definition 2.2. This generalized fractal-fractional integral replaces the Riemann–Liouville fractional integral present in the AVIM iteration operator given in (18).

Then, the iterative components to (22) are obtained in the following manner,

$$\begin{cases} \eta_0 = \phi(\kappa, 0), \\ \eta_1 = \mathcal{A}[\eta_0], \\ \eta_2 = \mathcal{A}[\eta_0 + \eta_1], \\ \vdots \\ \eta_{k+1} = \mathcal{A}[\eta_0 + \eta_1 + \dots + \eta_k]. \end{cases} \quad (23)$$

Consequently, the required FFAVIM solution to (21) is

$$\phi(\kappa, \tau) = \sum_{k=0}^{\infty} \eta_k(\kappa, \tau) \quad (24)$$

In what follows, we derive the formula for obtaining the generalized fractal-fractional integral of a function that are needed to compute the iterative components of the FFAVIM iterative operator, $\mathcal{A}[\phi]$.

Lemma 4.1. Given Definition 2.2, then,

$$I_t^{\alpha, \beta, \gamma} t^k = \frac{\Gamma\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)}{\beta^\alpha \Gamma\left(1 + \alpha + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)} t^{\alpha\beta + k + \gamma - 1}. \quad (25)$$

Proof. Using the definition in Definition 2.2, we have

$$I_t^{\alpha, \beta, \gamma} t^k = \frac{\gamma \beta^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \tau^{\beta + \gamma - 2} (t^\beta - \tau^\beta)^{\alpha-1} \tau^k d\tau. \quad (26)$$

After some mathematical simplifications involving change of variable, we obtain,

$$I_t^{\alpha, \beta, \gamma} t^k = \frac{\gamma \beta^{1-\alpha}}{\beta \Gamma(\alpha)} t^{\alpha\beta + \gamma + k - 1} \int_0^1 v^{\frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}} (1-v)^{\alpha-1} dv \quad (27)$$

$$= \frac{\gamma}{\beta^\alpha \Gamma(\alpha)} t^{\alpha\beta + \gamma + k - 1} B\left(\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right), \alpha\right) \quad (28)$$

$$= \frac{\gamma \Gamma\left(1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)}{\beta^\alpha \Gamma\left(1 + \alpha + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}\right)} t^{\alpha\beta + \gamma + k - 1}, \quad (29)$$

which completes the proof, where $B(p, q)$, $p = 1 + \frac{\gamma}{\beta} + \frac{k}{\beta} - \frac{1}{\beta}$, $q = \alpha$, represents the Beta function. $\square$

Remark 4.1. The generalized fractal-fractional integral of $u(x, t)$ in the case when $u(x, t)$ is a transcendental function can be obtained similarly by writing it in terms of series of t^k . For example, $e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$.

The FFAVIM procedure presented above is summarized in Algorithm 1 below,

Algorithm 1 Algorithm for FFAVIM solver.

Start algorithm

- 1: Define the FFPDE:

$$CD_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa, \tau) + L\varphi(\kappa, \tau) + N\varphi(\kappa, \tau) = g(\kappa, \tau),$$

where $CD_{\tau}^{\alpha,\beta,\gamma}$ is the NGCFFD operator, L and N are linear and nonlinear operators respectively, and $g(\kappa, \tau)$ is the source term.

- 2: Define the initial condition: $\varphi(\kappa, 0)$.
 3: Construct the FFAVIM iterative operator:

$$\mathcal{A}[\varphi] = I_{\tau}^{\alpha,\beta,\gamma} \left[\frac{(-1)^m}{(m-1)!} (s-\tau)^{m-1} \left(CD_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa, \tau) + L\varphi(\kappa, \tau) + N\varphi(\kappa, \tau) - g(\kappa, \tau) \right) \right],$$

where $I_{\tau}^{\alpha,\beta,\gamma}$ is the new generalized fractal-fractional integral.

- 4: Compute the iterative approximations:

- Set initial guess:

$$\eta_0 = \varphi(\kappa, 0).$$

- Compute iteratively:

$$\begin{aligned} \eta_1 &= \mathcal{A}[\eta_0], \\ \eta_2 &= \mathcal{A}[\eta_0 + \eta_1], \\ \eta_3 &= \mathcal{A}[\eta_0 + \eta_1 + \eta_2], \\ &\vdots \\ \eta_{k+1} &= \mathcal{A}[\eta_0 + \eta_1 + \cdots + \eta_k]. \end{aligned}$$

- 5: Compute the final FFAVIM solution:

$$\varphi(\kappa, \tau) = \sum_{k=0}^{\infty} \eta_k(\kappa, \tau).$$

Output: Approximate solution $\varphi(\kappa, \tau)$ for the given FPDE.

4.1 Application of FFAVIM to diffusion-wave equation

This section presents the application of the proposed FFAVIM technique, using the NGCFFD to solve diffusion-wave equations.

Consider the time-fractal-fractional diffusion-wave (FFD-W) equation as,

$${}^C\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}\varphi(\kappa, \tau) = c\varphi_{\kappa\kappa}(\kappa, \tau) + f(\kappa), \quad (30)$$

with $\varphi(0, \tau) = \varphi(L, \tau) = 0$, $\varphi(\kappa, 0) = \phi(\kappa)$, $\varphi_{\tau}(\kappa, 0) = \sigma(\kappa)$, $0 \leq \kappa \leq L$, $\tau > 0$ and ${}^C\mathfrak{D}_{\tau}^{\alpha,\beta,\gamma}$ is the NGCFFD.

To obtain the solution of (30) by FFAVIM method, we define the iterative operator, $\mathcal{A}[\phi]$ as,

$$\mathcal{A}[\phi(\kappa, \tau)] = I_{\tau}^{\alpha, \beta, \gamma} \left[\frac{(-1)^m}{(m-1)!} (s-\tau)^{m-1} \left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \phi(\kappa, \tau) - c \phi_{\kappa\kappa}(\kappa, \tau) - f(\kappa) \right) \right]. \quad (31)$$

The iterative components are then computed using (23). The FFAVIM approximation would then be obtained in series form as in (24).

4.2 Application of FFAVIM to generalized fractional PDE with variable coefficients

Here, we demonstrate how our proposed FFAVIM technique can be used to solve time-FFPDEs having variable coefficients in the sense of NGCFFD.

Consider the generalized non-linear time-fractal-fractional PDE with variable coefficients as,

$$\begin{aligned} {}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \phi(\kappa, \tau) - \mu(\kappa) \phi_{\kappa\kappa\tau}(\kappa, \tau) + 2\varphi(\kappa) \phi_{\kappa}(\kappa, \tau) + \omega(\kappa) \phi(\kappa, \tau) \phi_{\kappa}(\kappa, \tau) \\ - \theta(\kappa) \phi(\kappa, \tau) \phi_{\kappa\kappa\kappa}(\kappa, \tau) - \sigma(\kappa) \phi_{\kappa}(\kappa, \tau) \phi_{\kappa\kappa}(\kappa, \tau) = 0, \quad \ell < \kappa < \mathcal{L}, \quad \ell, \mathcal{L} \in \mathbb{R}, \end{aligned} \quad (32)$$

with the initial and BCs,

$$\phi(\kappa, 0) = \phi(\kappa), \quad \phi(\ell, \tau) = 0, \quad \phi(\mathcal{L}, \tau) = 0, \quad \tau \geq 0,$$

where ${}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma}$ is the NGCFFD.

To obtain the solution of (32) using FFAVIM method, the iterative operator $\mathcal{A}[\phi]$ for (32) is given as,

$$\begin{aligned} \mathcal{A}[\phi(\kappa, \tau)] = I_{\tau}^{\alpha, \beta, \gamma} \left[\frac{(-1)^m}{(m-1)!} (s-\tau)^{m-1} \left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \phi(\kappa, \tau) - \mu(\kappa) \phi_{\kappa\kappa\tau}(\kappa, \tau) + 2\varphi(\kappa) \phi_{\kappa}(\kappa, \tau) \right. \right. \\ \left. \left. + \omega(\kappa) \phi(\kappa, \tau) \phi_{\kappa}(\kappa, \tau) - \theta(\kappa) \phi(\kappa, \tau) \phi_{\kappa\kappa\kappa}(\kappa, \tau) - \sigma(\kappa) \phi_{\kappa}(\kappa, \tau) \phi_{\kappa\kappa}(\kappa, \tau) \right) \right]. \end{aligned} \quad (33)$$

The iterative components would be computed next with the help of (23) to obtain the series solution of the form in (24).

5 Numerical Examples

In this part, the efficiency of the proposed technique is demonstrated using some numerical examples.

Example 5.1. Consider the FFD-W equation as given in [53],

$$\begin{aligned} {}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \varphi(\kappa, \tau) = \varphi_{\kappa\kappa}(\kappa, \tau), \quad 0 < \alpha \leq 1, \\ \varphi(0, \tau) = \varphi(\pi, \tau) = 0, \quad \varphi(\kappa, 0) = \sin \kappa, \quad \varphi_{\tau}(\kappa, 0) = 0, \quad 0 \leq \kappa \leq \pi, \quad \tau > 0. \end{aligned} \quad (34)$$

The exact solution to the classical model (when $\alpha = \beta = \gamma = 1$) is $\varphi(\kappa, \tau) = e^{-\tau} \sin \kappa$.

To solve (34) by our proposed FFAVIM, we construct the iteration formula and follow the FFAVIM algorithm as below,

$$\mathcal{A}[\varphi] = I_{\tau}^{\alpha, \beta, \gamma} \left[(-1) \left({}^C D_{\tau}^{\alpha, \beta, \gamma} \varphi(\kappa, \tau) - \varphi_{\kappa\kappa}(\kappa, \tau) \right) \right].$$

Then, the iterative components are given by,

$$\begin{cases} \eta_0 = \sin \kappa, \\ \eta_1 = -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \eta_0) - (\eta_0)_{\kappa\kappa} \right], \\ \vdots \\ \eta_{k+1} = -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} [\eta_0 + \eta_1 + \cdots + \eta_k]) - ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa\kappa} \right]. \end{cases} \quad (35)$$

We compute each component of (35) as follows,

$$\begin{aligned} \eta_1 &= -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \eta_0) - (\eta_0)_{\kappa\kappa} \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \sin \kappa) - (\sin \kappa)_{\kappa\kappa} \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \sin \kappa) + \sin \kappa \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} [0 + \sin \kappa] = -I_{\tau}^{\alpha, \beta, \gamma} [\sin \kappa \tau^0] = -\sin \kappa I_{\tau}^{\alpha, \beta, \gamma} [\tau^0], \end{aligned}$$

using Corollary 2.1, we get

$$\eta_1 = -\sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1}.$$

Similarly, for the second iterative component, we have

$$\begin{aligned} \eta_2 &= -I_{\tau}^{\alpha, \beta, \gamma} \left[({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} [\eta_0 + \eta_1]) - ([\eta_0 + \eta_1])_{\kappa\kappa} \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} \left[\left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \left[\sin \kappa - \sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \right] \right) \right. \\ &\quad \left. - \left(\left[\sin \kappa - \sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \right] \right)_{\kappa\kappa} \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} \left[\left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \left[-\sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \right] \right) \right. \\ &\quad \left. - \left(\left[-\sin \kappa + \sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \right] \right) \right], \end{aligned}$$

and with the aid of Theorem 2.1 and Corollary 2.1, we obtain

$$= -\sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^\alpha \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \\ - I_{\tau}^{\alpha, \beta, \gamma} \left(\left[-\sin \kappa + \sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^\alpha \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1} \right] \right),$$

which simplifies to

$$\eta_2 = \sin \kappa \frac{\gamma^2 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)}{\beta^{2\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)} \tau^{2\alpha\beta + 2\gamma - 2}.$$

Higher-order iterative components can be computed by following a similar procedure. Thus, the following successive approximations are obtained for the iterative components for Example 5.1 up to the third order,

$$\eta_0 = \sin \kappa,$$

$$\eta_1 = -\sin \kappa \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^\alpha \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \tau^{\alpha\beta + \gamma - 1},$$

$$\eta_2 = \sin \kappa \frac{\gamma^2 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)}{\beta^{2\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)} \tau^{2\alpha\beta + 2\gamma - 2}, \quad (36)$$

$$\eta_3 = \sin \kappa \frac{\gamma^3 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta} \right)}{\beta^{3\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right)} \tau^{3\alpha\beta + 3\gamma - 3},$$

⋮

Table 1 shows the result and absolute errors of Example 5.1 using only three levels of FFAVIM iterations when $\alpha = 1$, $\beta = 1.014$, $\gamma = 0.98$. It is apparent that we do not require many iterations (only three) to obtain good approximation to the exact value. The outcome displayed in Table 2 is the results of using FFAVIM for Example 5.1 with different α values: $\alpha = 0.5, 0.75$ and constant β and γ values.

Table 1: Approximate solution of Example 5.1 with third-order FFAVIM for $\alpha = 1, \beta = 1.014, \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM approximate	Absolute error
0.1	0.1	0.09033301095	0.09033593761	2.926660 E-6
	0.2	0.08173668840	0.08180429106	6.760266 E-5
	0.3	0.07395841409	0.07408157185	1.231577 E-4
	0.4	0.06692034044	0.06705706090	1.367205 E-4
	0.5	0.06055202806	0.06062786565	7.583759 E-5
	0.6	0.05478974073	0.05469444280	9.529793 E-5
	0.7	0.04957580754	0.04915911386	4.166937 E-5
	0.8	0.04485804569	0.04392539906	9.326466 E-4
	0.9	0.04058923823	0.03889766300	1.691575 E-3
	1.0	0.03672666153	0.03398090423	2.745757 E-3
0.2	0.1	0.17976344430	0.17976926840	1.582410 E-6
	0.2	0.16265669080	0.16279122060	1.345298 E-4
	0.3	0.14717786020	0.14742294510	2.450849 E-4
	0.4	0.13317203500	0.13344410990	2.720749 E-4
	0.5	0.12049904030	0.12064995770	1.509174 E-4
	0.6	0.10903204050	0.10884239670	1.896438 E-4
	0.7	0.09865626990	0.09782704607	8.292239 E-4
	0.8	0.08926788460	0.08741190995	1.855975 E-3
	0.9	0.08077292220	0.07740667346	3.366249 E-3
	1.0	0.07308636240	0.06762228247	5.464079 E-3

Table 2: Absolute errors of third-order FFAVIM solution of Example 5.1 for $\kappa = 0.1$ and different values of $\alpha, \beta = 1.014, \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM $\alpha = 0.5$	Abs. Error $\alpha = 0.5$	FFAVIM $\alpha = 0.75$	Abs. Error $\alpha = 0.75$
0.1	0.1	0.09033301095	0.07148829286	1.884471809E-2	0.08255739134	7.77561961E-3
	0.2	0.08173668840	0.06253119485	1.920549355E-2	0.07299952570	8.73716270E-3
	0.3	0.07395841409	0.05571294351	1.824547058E-2	0.06561837954	8.34003455E-3
	0.4	0.06692034044	0.04964625123	1.727408921E-2	0.05939933823	7.52100221E-3
	0.5	0.06055202806	0.04384532392	1.670670414E-2	0.05385047621	6.70155185E-3
	0.6	0.05478974073	0.03808519375	1.670454698E-2	0.04867098277	6.11875796E-3
	0.7	0.04957580754	0.03224594660	1.732986094E-2	0.04365320398	5.92260356E-3
	0.8	0.04485804569	0.02625814293	1.859990276E-2	0.03864257334	6.21547235E-3
	0.9	0.04058923823	0.02007946475	2.050977348E-2	0.03351797086	7.07126737E-3
	1.0	0.03672666153	0.01368334036	2.304332117E-2	0.02818088578	8.54577575E-3

Figure 1 is a comparison of the FFAVIM solution of Example 5.1 with the exact solution of the classical model ($\alpha = 1$). The effectiveness of the method is clear from the two graphical simulations. While Figure 2 represents the surface plot of the FFAVIM solution of Example 5.1 for two different α values. The effectiveness of the FFAVIM can be visualized from these plots.

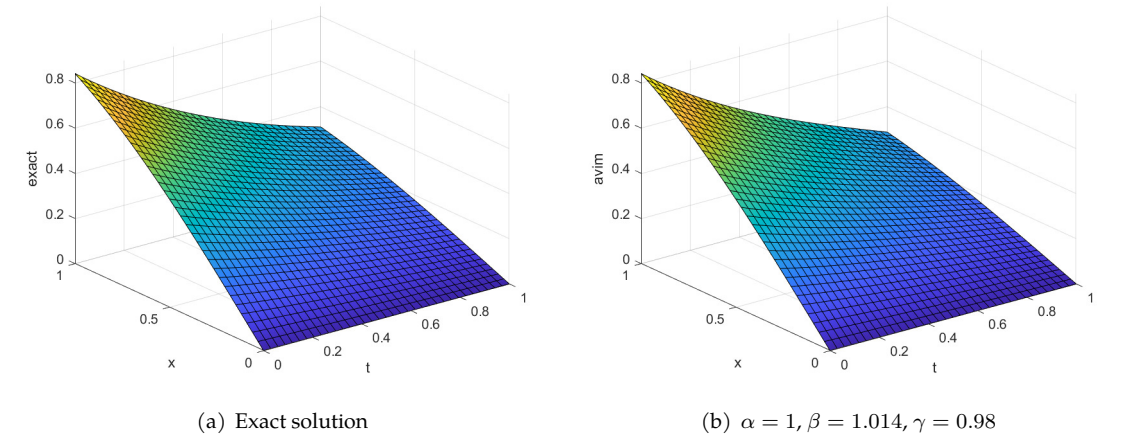


Figure 1: Graphical simulation of third-order FFAVIM for $\alpha = 1$ and exact solutions of Example 5.1.

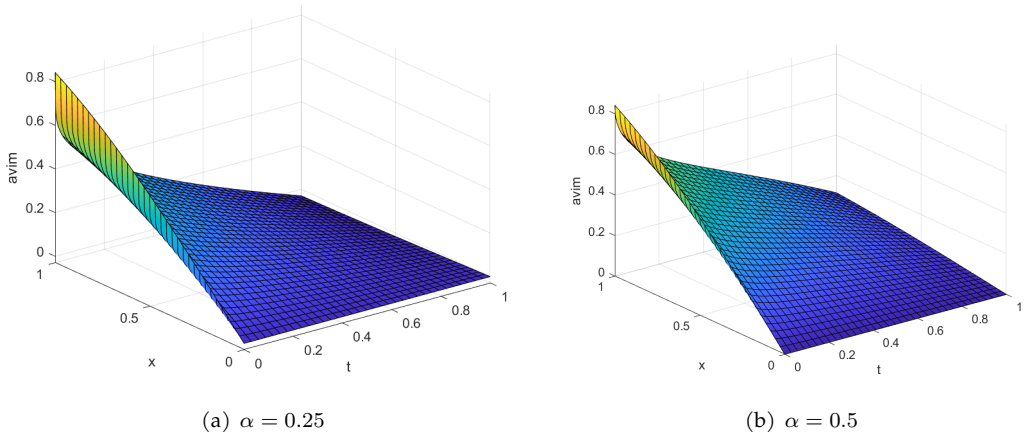


Figure 2: Graphical simulation of third-order FFAVIM solution of Example 5.1 for different α values, $\beta = 1.014$, $\gamma = 0.98$.

Figure 3 shows the effect of varying the fractional parameter α which defines memory effect. Although, as α decreases, convergence to the exact solution decreases, the FFAVIM is still able to provide suitable approximate solution to the problem.

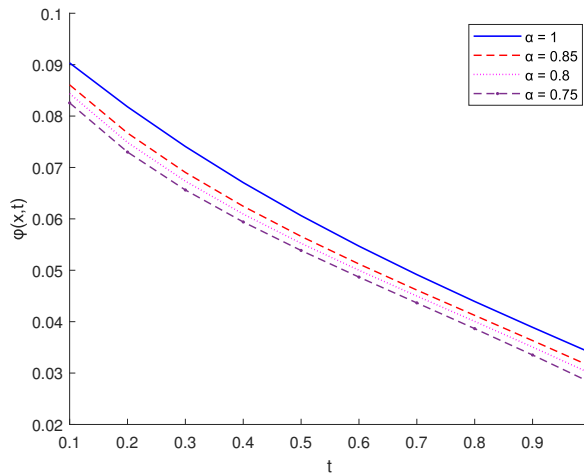


Figure 3: Plots of third-order FFAVIM solution of Example 5.1 for different α values with $\beta = 1.014$, $\gamma = 0.98$

Example 5.2. Consider the fractal-fractional PDE with constant coefficients in the sense of NGCFFD [54],

$${}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\phi(\kappa,\tau) - \phi_{\kappa\kappa\tau}(\kappa,\tau) + \phi_\kappa(\kappa,\tau) + \phi(\kappa,\tau)\phi_\kappa(\kappa,\tau) - \phi(\kappa,\tau)\phi_{\kappa\kappa\kappa}(\kappa,\tau) - 3\phi_\kappa(\kappa,\tau)\phi_{\kappa\kappa}(\kappa,\tau) = 0, \quad 0 < \tau \leq 1, \quad 0 < \alpha \leq 1, \quad (37)$$

with initial condition: $\phi(\kappa, 0) = \frac{4}{3}e^{\frac{1}{2}\kappa}$.

Equation (37) is the well-known fractional order Fornberg–Whitham equation, an important equation modelling wave breakage in mathematical physics. The exact travelling wave solution to (37) (classical model) is $\phi(\kappa, \tau) = \frac{4}{3}e^{\frac{1}{2}\kappa - \frac{2}{3}\tau}$.

To obtain its solution by our proposed FFAVIM, the iteration formula is equally constructed and the FFAVIM algorithm is utilized to complete the solution process as presented below,

$$\mathcal{A}[\phi] = I_\tau^{\alpha,\beta,\gamma} \left[(-1) \left({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\phi(\kappa,\tau) - \phi_{\kappa\kappa\tau}(\kappa,\tau) + \phi_\kappa(\kappa,\tau) + \phi(\kappa,\tau)\phi_\kappa(\kappa,\tau) - \phi(\kappa,\tau)\phi_{\kappa\kappa\kappa}(\kappa,\tau) - 3\phi_\kappa(\kappa,\tau)\phi_{\kappa\kappa}(\kappa,\tau) \right) \right],$$

with iterative components,

$$\left\{ \begin{array}{l} \eta_0 = \frac{4}{3}e^{\frac{1}{2}\kappa}, \\ \eta_1 = -I_\tau^{\alpha,\beta,\gamma} \left[({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}\eta_0) - (\eta_0)_{\kappa\kappa\tau} + (\eta_0)_\kappa + \eta_0(\eta_0)_\kappa - \eta_0(\eta_0)_{\kappa\kappa\kappa} - 3(\eta_0)_\kappa(\eta_0)_{\kappa\kappa} \right], \\ \vdots \\ \eta_{k+1} = -I_\tau^{\alpha,\beta,\gamma} \left[({}^C\mathfrak{D}_\tau^{\alpha,\beta,\gamma}[\eta_0 + \eta_1 + \cdots + \eta_k]) - ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa\kappa\tau} \right. \\ \quad + ([\eta_0 + \eta_1 + \cdots + \eta_k])_\kappa + [\eta_0 + \eta_1 + \cdots + \eta_k]([\eta_0 + \eta_1 + \cdots + \eta_k])_\kappa \\ \quad - [\eta_0 + \eta_1 + \cdots + \eta_k]([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa\kappa\kappa} \\ \quad \left. - 3([\eta_0 + \eta_1 + \cdots + \eta_k])_\kappa([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa\kappa} \right] \end{array} \right. \quad (38)$$

We derive the components of (38) as follows,

$$\begin{aligned}\eta_1 &= -I_{\tau}^{\alpha, \beta, \gamma} \left[\left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \eta_0 \right) - (\eta_0)_{\kappa \kappa \tau} + (\eta_0)_{\kappa} + \eta_0 (\eta_0)_{\kappa} - \eta_0 (\eta_0)_{\kappa \kappa \kappa} - 3(\eta_0)_{\kappa} (\eta_0)_{\kappa \kappa} \right] \\ &= -I_{\tau}^{\alpha, \beta, \gamma} \left[\left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \frac{4}{3} e^{\frac{1}{2} \kappa} \right) - \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa \kappa \tau} + \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa} + \frac{4}{3} e^{\frac{1}{2} \kappa} \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa} \right. \\ &\quad \left. - \frac{4}{3} e^{\frac{1}{2} \kappa} \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa \kappa \kappa} - 3 \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa} \left(\frac{4}{3} e^{\frac{1}{2} \kappa} \right)_{\kappa \kappa} \right].\end{aligned}$$

On simplification, we get,

$$\eta_1 = -\frac{2}{3} e^{\frac{1}{2} \kappa} \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \tau^{\alpha \beta + \gamma - 1}}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}.$$

In a similar vein, we can obtain the third- and higher-order approximations by following the same procedure. Consequently, successive approximations up to the third order are given as,

$$\eta_0 = \frac{4}{3} e^{\frac{1}{2} \kappa},$$

$$\eta_1 = -\frac{2}{3} e^{\frac{1}{2} \kappa} \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \tau^{\alpha \beta + \gamma - 1}}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)},$$

$$\begin{aligned}\eta_2 &= \frac{1}{3} e^{\frac{1}{2} \kappa} \frac{\gamma^2 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{2\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \left(\frac{\Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \tau^{2\alpha \beta + 2\gamma - 2}}{\Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)} \right. \\ &\quad \left. - \frac{1}{2} \frac{(\alpha \beta + \gamma - 1) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \tau^{2\alpha \beta + 2\gamma - 3}}{\Gamma} \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \right),\end{aligned}$$

$$\begin{aligned}\eta_3 &= \frac{1}{6} e^{\frac{1}{2} \kappa} \frac{\gamma^2 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{2\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \left(-\frac{\gamma \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta} \right) \tau^{3\alpha \beta + 3\gamma - 3}}{\beta \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta} \right)} \right. \\ &\quad - \frac{1}{4} \frac{\gamma (\alpha \beta + \gamma - 1) (2\alpha \beta + 2\gamma - 3) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{5}{\beta} \right) \tau^{3\alpha \beta + 3\gamma - 5}}{\beta \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \Gamma \left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{5}{\beta} \right)} \\ &\quad + \frac{1}{2} \frac{\gamma (2\alpha \beta + 2\gamma - 2) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{4}{\beta} \right) \tau^{3\alpha \beta + 3\gamma - 4}}{\beta \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \Gamma \left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{4}{\beta} \right)} \\ &\quad \left. + \frac{1}{2} \frac{\gamma (\alpha \beta + \gamma - 1) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{4}{\beta} \right) \tau^{3\alpha \beta + 3\gamma - 4}}{\beta \Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \Gamma \left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{4}{\beta} \right)} \right),\end{aligned}$$

⋮

(39)

Similar results as in Table 1 are observed in Table 3 where we show the result and absolute errors of Example 5.2 when $\alpha = 1, \beta = 1.014, \gamma = 0.98$. The FFAVIM gives a good approximation in terms of absolute errors with fewer terms needed in the series solution.

Table 3: Approximate solution of Example 5.2 with third-order FFAVIM for $\alpha = 1, \beta = 1.014, \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM approximate	Absolute error
−4	0.02	0.1780570525	0.1780639651	6.91260 E-6
	0.04	0.1756987155	0.1757334374	3.47219 E-5
	0.06	0.1733716145	0.1734389308	6.73163 E-5
	0.08	0.1710753356	0.1710753356	1.01402 E-4
−2	0.02	0.4840092501	0.4840280409	1.87908 E-5
	0.04	0.4775986256	0.4776930098	9.43842 E-5
	0.06	0.4712729093	0.4714558942	1.82985 E-4
	0.08	0.4651219777	0.4653066161	2.75640 E-4
2	0.02	3.5763715010	3.5765103470	1.38846 E-4
	0.04	3.5290030390	3.5297004470	6.97408 E-4
	0.06	3.4822619640	3.4836140490	1.35209 E-3
	0.08	3.4361399680	3.4381766880	2.03672 E-3
4	0.02	9.7215856650	9.7219630870	3.77422 E-4
	0.04	9.5928248290	9.5937699020	9.45073 E-4
	0.06	9.4657694200	9.5947205850	1.89576 E-3
	0.08	9.3443799560	9.3459332170	5.53638 E-3

In Table 4, we present the result of solving Example 5.2 with varying value of the fractal parameter, β taken between 1.00 and 1.01 with $\alpha = 1$. The results are also compared with those obtained using the method in [54]. It is observed that the accuracy of FFAVIM improves just by varying the parameter β , with time t , which shows that the extra parameter β affords us the chance to obtain better results without increasing the number of terms in the iterative process that would have led to more computations. In addition, our proposed method also outperforms the method in [54] utilizing more number of iterations (five) compared to that of our method (three).

Table 4: Comparison of absolute errors between third-order FFAVIM and fifth-order AVIM in [54] for Example 5.2 for $\alpha = 1, \beta \in [1.00, 1.01], \gamma = 0.98$.

κ	τ	Exact solution	FFAVIM	AVIM in [54]	Abs. error of FFAVIM	Abs. error of AVIM in [54]
0.5	0.2	1.49832638	1.498326756	1.49837370	3.7500 E-7	4.7320 E-5
	0.4	1.31129527	1.311295782	1.31123148	5.1000 E-7	6.3790 E-5
	0.6	1.14761064	1.147614708	1.14746139	4.0732 E-6	1.4924 E-4
	0.8	1.00435820	1.004339344	1.00422572	1.8865 E-5	1.3248 E-4
	1.0	0.87898750	0.878993035	0.87897781	5.5279 E-6	9.6900 E-6
1.0	0.2	1.92388915	1.923889637	1.92394991	4.8101 E-7	6.0760 E-5
	0.4	1.68373645	1.683737113	1.68365456	6.5553 E-7	8.1890 E-5
	0.6	1.47356122	1.473566454	1.47336959	5.2301 E-6	1.9163 E-4
	0.8	1.28962146	1.289597245	1.28945134	2.4222 E-5	1.7012 E-4
	1.0	1.12864229	1.128649398	1.12862985	7.0980 E-6	1.2440 E-5

Similar outcome is observed in Table 5 where we present the results of applying our proposed FFAVIM method in comparison with the AVIM method in [54] to solve Example 5.2 with $\alpha = 0.9$, $\beta = 1.08$ and $\gamma = 0.98$. The FFAVIM method competes favourably with the AVIM despite being two iteration terms less. This further confirms the importance of the extra parameter present in NGCFFD together with the fractal dimension utilized in FFAVIM procedure.

Table 5: Comparison of absolute errors between third-order FFAVIM and fifth-order AVIM in [54] for Example 5.2, $\alpha = 0.9$, $\beta = 1.08$, $\gamma = 0.98$.

κ	τ	Exact solution	FFAVIM	AVIM in [54]	Abs. error FFAVIM	Abs. error of AVIM in [54]
0.5	0.2	1.49832638	1.4901580790	1.4405122780	8.168302 E-3	5.7814102 E-2
	0.4	1.31129527	1.3132056820	1.2556273427	1.910410 E-3	5.5667927 E-2
	0.6	1.14761064	1.1639177080	1.1079229036	1.630707 E-3	3.9687736 E-2
	0.8	1.00435820	1.0369507060	0.9856303461	3.259250 E-2	1.8727850 E-2
	1.0	0.87898750	0.9287880425	0.8823294271	4.980054 E-2	3.3419270 E-3
1.0	0.2	1.92388915	1.9134008500	1.8496543780	1.048831 E-2	7.4234772 E-2
	0.4	1.68373645	1.6861894710	1.6122574219	2.453014 E-2	7.1479028 E-2
	0.6	1.47356122	1.4944999190	1.4226011679	2.093870 E-2	5.0960052 E-2
	0.8	1.28962146	1.3314710640	1.2655744159	4.184960 E-2	2.4047044 E-2
	1.0	1.12864229	1.1925874540	1.1329334103	6.394515 E-2	4.2911200 E-3

Figures 4–6 visualizes the behaviour of the FFAVIM solution of Example 5.2 for different α values when contrasted with that of the exact solution. It is clear that the proposed method gives graphical simulations which mimics that of the exact solution, confirming its suitability.

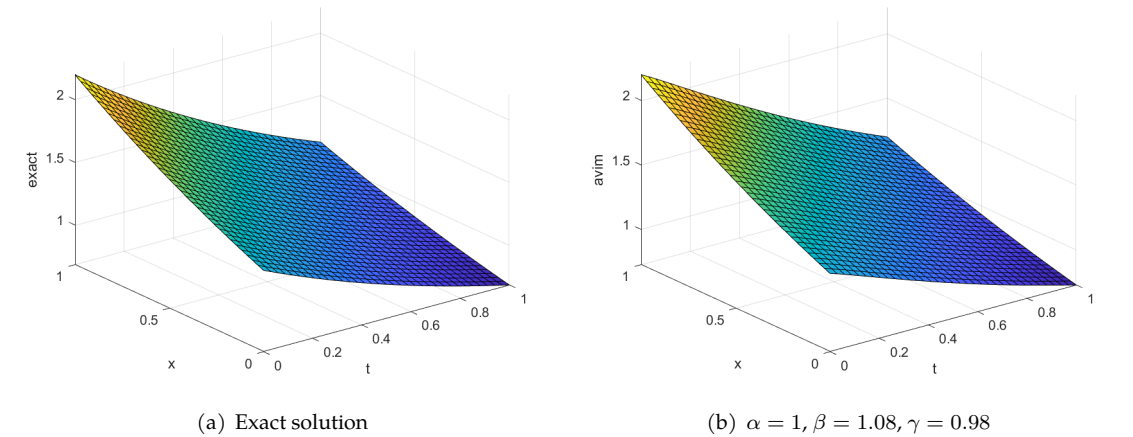


Figure 4: Graphical simulation of third-order FFAVIM and exact solutions of Example 5.2.

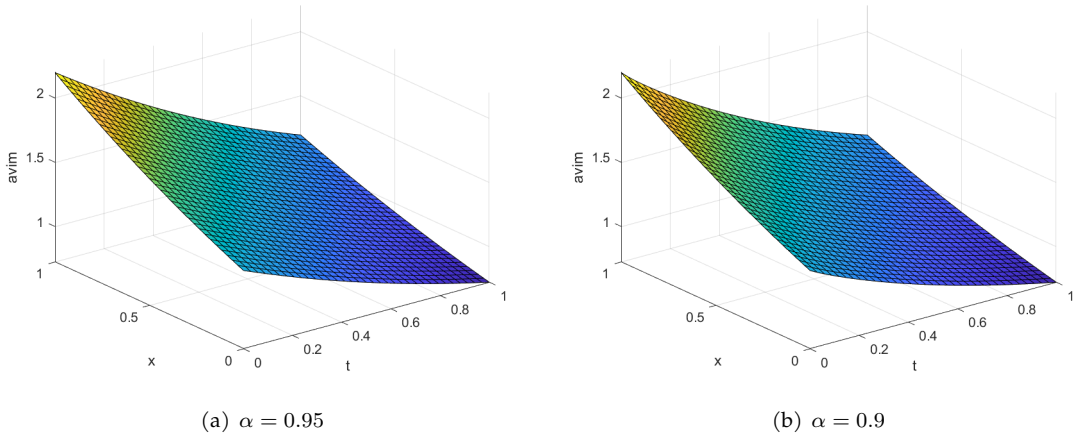


Figure 5: Graphical simulation of third-order FFAVIM solution of Example 5.2 for different values of $\alpha, \beta = 1.08, \gamma = 0.98$.

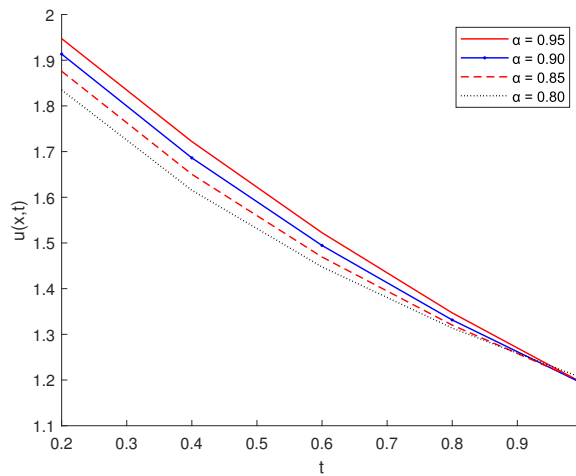


Figure 6: Plots of third-order FFAVIM solution of Example 5.2 for different values of α with $\beta = 1.08, \gamma = 0.98$.

Example 5.3. Consider the fractal-fractional PDE with variable coefficients [5],

$${}^C \mathfrak{D}_\tau^{\alpha, \beta, \gamma} \phi - \kappa^2 \phi_{\kappa \kappa \tau} + 2\phi_\kappa + \phi\phi - \phi\phi_{\kappa \kappa \kappa} - 3\phi_\kappa \phi_{\kappa \kappa} = 0, \quad 0 < \tau \leq 1, \quad 0 < \alpha \leq 1. \quad (40)$$

with initial condition: $\phi(\kappa, 0) = \kappa^2$. The exact solution to the classical model of the given FFPDE is not presented in [5].

The iteration formula to (40) using the FFAVIM method is equally obtained by means of (33) and its iterative components also computed as below,

$$\mathcal{A}[\phi] = I_\tau^{\alpha, \beta, \gamma} \left[(-) \left({}^C \mathfrak{D}_\tau^{\alpha, \beta, \gamma} \phi - \kappa^2 \phi_{\kappa \kappa \tau} + 2\phi_\kappa + \phi\phi - \phi\phi_{\kappa \kappa \kappa} - 3\phi_\kappa \phi_{\kappa \kappa} \right) \right],$$

whose iterative components are

$$\left\{ \begin{array}{l} \eta_0 = \kappa^2, \\ \eta_1 = -I_{\tau}^{\alpha, \beta, \gamma} \left[\left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} \eta_0 \right) - \kappa^2 (\eta_0)_{\kappa \kappa \tau} + 2 (\eta_0)_{\kappa} + \eta_0 (\eta_0)_{\kappa} - \eta_0 (\eta_0)_{\kappa \kappa \kappa} - (\eta_0)_{\kappa} (\eta_0)_{\kappa \kappa} \right], \\ \vdots \\ \eta_{k+1} = -I_{\tau}^{\alpha, \beta, \gamma} \left[\begin{array}{l} \left({}^C \mathfrak{D}_{\tau}^{\alpha, \beta, \gamma} [\eta_0 + \eta_1 + \cdots + \eta_k] \right) - \kappa^2 ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa \kappa \tau} \\ + 2 ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa} + [\eta_0 + \eta_1 + \cdots + \eta_k] ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa} \\ - [\eta_0 + \eta_1 + \cdots + \eta_k] ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa \kappa \kappa} \\ - 3 ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa} ([\eta_0 + \eta_1 + \cdots + \eta_k])_{\kappa \kappa} \end{array} \right]. \end{array} \right. \quad (41)$$

Again, applying the FFAVIM algorithm in a manner analogous to the previous examples, we obtain the following successive approximations up to the second order for this example,

$$\begin{aligned} \eta_0 &= \kappa^2, \\ \eta_1 &= -(2\kappa^3 - 8\kappa) \frac{\gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \tau^{\alpha \beta + \gamma - 1}}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}, \\ \eta_2 &= \frac{-12\gamma^2 \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)}{\beta^{2\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right)} \left(\frac{\kappa^3 (\alpha \beta + \gamma - 1) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right) \tau^{2\alpha \beta + 2\gamma - 3}}{\Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{3}{\beta} \right)} \right. \\ &\quad - \frac{1}{6} \frac{(5\kappa^4 - 66\kappa^2 + 16) \Gamma \left(1 + \alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right) \tau^{2\alpha \beta + 2\gamma - 2}}{\Gamma \left(1 + 2\alpha + \frac{2\gamma}{\beta} - \frac{2}{\beta} \right)} \\ &\quad \left. + \frac{1}{6} \frac{(6\kappa^5 - 152\kappa^3 + 224\kappa) \gamma \Gamma \left(1 + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + 2\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta} \right) \tau^{3\alpha \beta + 3\gamma - 3}}{\beta^{\alpha} \Gamma \left(1 + \alpha + \frac{\gamma}{\beta} - \frac{1}{\beta} \right) \Gamma \left(1 + 3\alpha + \frac{3\gamma}{\beta} - \frac{3}{\beta} \right)} \right), \\ &\vdots \end{aligned} \quad (42)$$

In Tables 6 and 7, we display the results of approximating the variable-coefficient FFPDE in Example 5.3 with a second-order FFAVIM at different α values. The numerical values are tallied alongside those of the traditional VIM in [5] proposed for the same problem. The exact solution isn't available, however, the correctness of the proposed FFAVIM is affirmed via the comparison of the difference between the FFAVIM solution and that of VIM.

Table 6: Comparison of second-order FFAVIM solution of Example 5.3 with VIM in [5] for $\alpha = 1, \beta = 0.98, \gamma = 0.98$.

κ	τ	FFAVIM	VIM in [5]	FFAVIM-VIM
0.1	0.1	0.2895464737	0.3414716667	5.1925193 E-2
	0.2	0.9047899428	0.9577533333	5.2963391 E-2
	0.3	1.8354881748	1.8768450000	4.1356825 E-2
	0.4	3.0696448141	3.0967466667	2.7101853 E-2
	0.5	4.5984369297	4.6154583333	1.7021404 E-2
	0.6	6.4147424850	6.4309800000	1.6237515 E-2
	0.7	8.5125052282	8.5413116667	2.8806438 E-2
	0.8	10.886400921	10.944453333	5.8052412 E-2
0.2	0.1	0.3722695731	0.5605466667	1.8827709 E-1
	0.2	0.9924681385	1.2664533333	2.7398519 E-1
	0.3	1.8854063834	2.2357200000	3.5031362 E-1
	0.4	3.0409321041	3.4663466667	4.2541456 E-1
	0.5	4.4513724066	4.9563333333	5.0496093 E-1
	0.6	6.1104349749	6.7036800000	5.9324503 E-1
	0.7	8.0127093689	8.7063866667	6.9367730 E-1
	0.8	10.153398232	10.962453333	8.0905510 E-1

Table 7: Comparison of second-order FFAVIM solution of Example 5.3 with VIM in [5] for $\alpha = 0.75, \beta = 0.98, \gamma = 0.98$.

κ	τ	FFAVIM	VIM in [5]	FFAVIM-VIM
0.1	0.1	1.049456121	1.060903465	1.1447344 E-2
	0.2	2.631479963	2.591106901	4.0373062 E-2
	0.3	4.558209979	4.477801402	8.0408577 E-2
	0.4	6.755030059	6.655117794	9.9912265 E-2
	0.5	9.178970821	9.083168720	9.5802101 E-2
	0.6	11.80119343	11.73432536	6.6868070 E-2
	0.7	14.60063609	14.58793515	1.2700940 E-2
	0.8	17.56105735	17.62777490	6.6717550 E-2
0.2	0.1	1.096826786	1.331249022	2.3442224 E-1
	0.2	2.581554361	2.884825181	3.0327082 E-1
	0.3	4.351547692	4.721597056	3.7004936 E-1
	0.4	6.347247222	6.794716853	4.4746963 E-1
	0.5	8.533567918	9.073961079	5.4039316 E-1
	0.6	10.88667099	11.53773303	6.5106204 E-1
	0.7	13.38900518	14.16957486	7.8056968 E-1
	0.8	16.02695688	16.95638418	9.2942730 E-1

Figure 7 visualizes the behavior of the FFAVIM solution of Example 5.3 in 3D graphical plot for different values of fractional parameter, α . Again, the suitability of the proposed approach is clear from the graphical simulations in Figures 7(a)–7(d) for $\alpha = 1, 0.9, 0.85$ and 0.80 respectively.

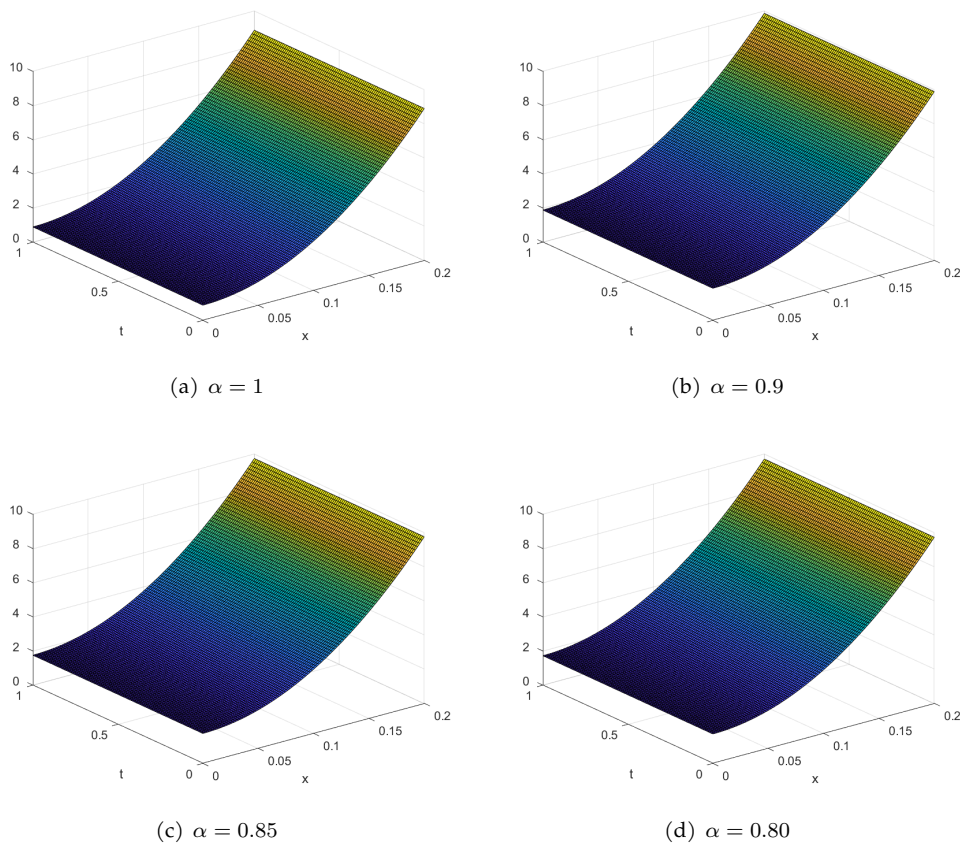


Figure 7: Graphical simulation of second-order FFAVIM solution of Example 5.3 for different α values, $\beta = 1.08, \gamma = 0.98$.

In Figure 8, further comparison is made between the plot of FFAVIM solution of Example 5.3 for different α values. It is apparent that the method is suitable for FFPDEs whose coefficients are not necessarily constants.

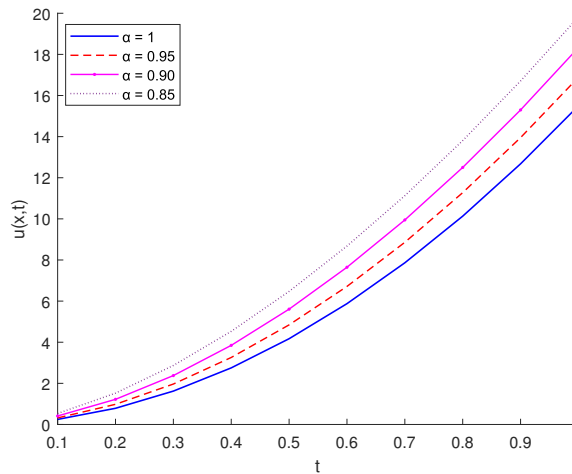


Figure 8: Plots of third-order FFAVIM solution of Example 5.3 for different α values with $\beta = 1.01, \gamma = 0.98$.

6 Conclusions

In this study, we introduced and successfully applied a modified Alternative Variational Iteration Method (FFAVIM) to approximate solutions of generalized time-fractal-fractional partial differential equations (FFPDEs) with both constant and variable coefficients. The fractal-fractional derivative was considered in the sense of the generalized Caputo-type fractal-fractional derivative (NGCFFD). The proposed method was implemented directly, without requiring linearization or perturbation, and provided analytical solutions in series form. To demonstrate the effectiveness of this technique, numerical examples are solved and results are presented in tables and graphs. The results indicate that accurate approximations can be achieved with fewer iterations of the FFAVIM, aided by the additional parameter and fractal dimension incorporated in the NGCFFD. This advantage distinguishes FFAVIM from other existing methods. Furthermore, the FFAVIM compares favorably with the classical AVIM [54] and the VIM [5], both of which require more iterative steps to attain similar accuracy. The findings suggest that the proposed method has strong potential for solving a broader class of real-world mathematical models beyond those explored in this study. Additionally, future research may focus on extending this approach to handle integro-differential equations and systems of fractal-fractional partial differential equations.

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Conflicts of Interest The authors declare that they have no conflict of interest.

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